ON THE SHARESHIAN-WACHS (q,r)-EULERIAN POLYNOMIALS

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Abstract. The (q,r)-Eulerian polynomials are the $(\mathrm{maj}-\mathrm{exc},\mathrm{fix},\mathrm{exc})$ enumerative polynomials of permutations. Using Shareshian and Wachs' exponential generating function of these Eulerian polynomials, Chung and Graham proved two symmetrical q-Eulerian identities and asked for bijective proofs. We provide such proofs using Foata and Han's three-variable statistic $(\mathrm{inv},\mathrm{pix},\mathrm{lec})$. We also prove a new recurrence formula for the (q,r)-Eulerian polynomials and study a q-analogue of Chung and Graham's restricted Eulerian numbers. In particular, we obtain a symmetrical identity for these restricted q-Eulerian numbers with a combinatorial proof.

1. Introduction

The *Eulerian polynomials* $A_n(t) := \sum_{k=0}^{n} A_{n,k}t^k$ are defined by the exponential generating function

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{(1-t)e^z}{e^{zt}-te^z}. \tag{1.1}$$

The coefficients $A_{n,k}$ are called *Eulerian numbers*. The Eulerian numbers arise in a variety of contexts in mathematics. Let $\mathfrak{S}_n$ denote the set of permutations of $[n] := \{1,2,\dots,n\}$. For each $\pi \in \mathfrak{S}_n$, a value i, $1 \leq i \leq n-1$, is an *excedance* (resp. *descent*) of π if $\pi(i) > i$ (resp. $\pi(i) > \pi(i+1)$). Denote by $\mathrm{exc}(\pi)$ and $\mathrm{des}(\pi)$ the number of excedances and descents of π, respectively. It is well-known that the Eulerian number $A_{n,k}$ counts permutations in $\mathfrak{S}_n$ with k descents (or k excedances), that is

$$A_n(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}\,\pi} = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{exc}\,\pi}.$$

The reader is referred to [9, 21] for some leisurely historical introductions of Eulerian polynomials and Eulerian numbers.

Several q-analogs of Eulerian polynomials with combinatorial meanings have been studied in the literature (see [3, 7, 22, 27]). Recall that the *major index*, $\mathrm{maj}(\pi)$, of a permutation $\pi \in \mathfrak{S}_n$ is the sum of all the descents of π, i.e., $\mathrm{maj}(\pi) := \sum_{\pi(i)>\pi(i+1)} i$ and the number of *fixed points* of π, $\mathrm{fix}(\pi)$, is defined by $\mathrm{fix}(\pi) := |\{1 \leq i \leq n : \pi(i) = i\}|$. Define the

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Shareshian-Wachs (q,r)-Eulerian polynomials $A_n(t,r,q)$ by the following (q,r)-extension of (1.1):

$$\sum_{n\geq 0} A_n(t,r,q)\frac{z^n}{(q;q)_n}=\frac{(1-t)e(rz;q)}{e(tz;q)-te(z;q)}, \tag{1.2}$$

where $(q;q)_n:=\prod_{i=1}^n(1-q^i)$ and $e(z;q)$ is the q-exponential function defined by $e(z;q):=\sum_{n\geq 0}\frac{z^n}{(q;q)_n}$. The following interpretation for $A_n(t,r,q)$ was given [22, 24]:

$$A_n(t,r,q):=\sum_{\pi\in\mathfrak{S}_n} t^{\operatorname{exc}\pi}r^{\operatorname{fix}\pi}q^{(\operatorname{maj}-\operatorname{exc})\,\pi}. \tag{1.3}$$

These polynomials have attracted the attention of several authors (cf. [10–12, 15–17, 19, 23, 25]).

Let $A_n(t,q)=A_n(t,1,q)$. Define the *q-Eulerian numbers* $A_{n,k}(q)$ and the *fix-version of q-Eulerian numbers* $A_{n,k}^{(j)}(q)$:

$$A_n(t,q)=\sum_k A_{n,k}(q)t^k \quad \text{and} \quad A_n(t,r,q)=\sum_{j,k}A_{n,k}^{(j)}(q)r^jt^k.$$

By (1.3), we have the following interpretations

$$A_{n,k}(q)=\sum_{\substack{\pi\in\mathfrak{S}_n\\ \operatorname{exc}\pi=k}} q^{(\operatorname{maj}-\operatorname{exc})\pi} \quad \text{and} \quad A_{n,k}^{(j)}(q)=\sum_{\substack{\pi\in\mathfrak{S}_n\\ \operatorname{exc}\pi=k\\ \operatorname{fix}\pi=j}} q^{(\operatorname{maj}-\operatorname{exc})\pi}. \tag{1.4}$$

Recall that the q-binomial coefficients ${n\brack k}_q$ are defined by ${n\brack k}_q:=\frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}$ for $0\leq k\leq n$, and ${n\brack k}_q=0$ if $k<0$ or $k>n$.

In [15], answering a question of Chung, Graham and Knuth [6], the following symmetrical q-Eulerian identity was proved:

$$\sum_{k\geq 1}{a+b\brack k}_q A_{k,a-1}(q)=\sum_{k\geq 1}{a+b\brack k}_q A_{k,b-1}(q), \tag{1.5}$$

where a,b are integers with $a,b\geq 1$. Besides a generating function proof using (1.2), a bijective proof of (1.5) was also given in [15]. Recently, through analytical arguments, Chung and Graham [5] derived from (1.2) the following two further symmetrical q-Eulerian identities:

$$\sum_{k\geq 1}(-1)^k{a+b\brack k}_q q^{\binom{a+b-k}{2}}A_{k,a}(q)=\sum_{k\geq 1}(-1)^k{a+b\brack k}_q q^{\binom{a+b-k}{2}}A_{k,b}(q), \tag{1.6}$$

$$\sum_{k\geq 1}{a+b+j+1\brack k}_q A_{k,a}^{(j)}(q)=\sum_{k\geq 1}{a+b+j+1\brack k}_q A_{k,b}^{(j)}(q), \tag{1.7}$$

where a,b,j are integers with $a,b\geq 1$ and $j\geq 0$, and asked for bijective proofs. One of our purposes is to provide such proofs using another interpretation of $A_n(t,r,q)$ due

to Foata and Han [11], which was already shown to be successful in the bijective proof of (1.5) in [15].

Chung and Graham [5] also studied the *restricted Eulerian number*, $B^{(j)}_{n,k}$, which is the number of permutations $\pi \in \mathfrak{S}_n$ with $\mathrm{des}(\pi) = k$ and $\pi(j) = n$. We define the *q-analogue of restricted Eulerian numbers* $B^{(j)}_{n,k}(q)$ by the exponential generating function:

$$\sum_{n\geq j} B^{(j)}_{n,k}(q)\frac{t^k z^{n-1}}{(q;q)_{n-1}} = \left(\frac{A_{j-1}(t,q)(qz)^{j-1}}{(q;q)_{j-1}}\right)\frac{e(tz;q)-te(tz;q)}{e(tz;q)-te(z;q)}, \tag{1.8}$$

and find the following q-analogue of Chung and Graham's symmetrical identity [5, Theorem 2].

Theorem 1. *Let a, b, j be integers with $a, b \geq 1$ and $j \geq 2$. Then*

$$\sum_{k\geq 1} \begin{bmatrix} a+b+1 \\ k-1 \end{bmatrix}_q B^{(j)}_{k,a}(q) = \sum_{k\geq 1} \begin{bmatrix} a+b+1 \\ k-1 \end{bmatrix}_q B^{(j)}_{k,b}(q). \tag{1.9}$$

We will give two proof of Theorem 1, using generating function and bijections, respectively. The generating function proof motives us to find a new recurrence formula for $A_n(t,r,q)$, while the bijective proof answers an open question of Chung and Graham [5] in the $q = 1$ case.

This paper is organized as follows. In section 2, we review some preliminaries about the three-variable statistic (inv, pix, lec) and give the bijective proofs of (1.6) and (1.7). In section 3, we first prove a new recurrence formula for $A_n(t,r,q)$. We then define a new statistic called "rix", which together with descents and *admissible inversions* (a statistic on permutations appears in the context of poset topology [22]) gives another interpretation of $A_n(t,r,q)$. In section 4, we give two combinatorial interpretations of $B^{(j)}_{n,k}(q)$ and two proofs of Theorem 1.

2. Bijective proofs of (1.6) and (1.7)

2.1. **Preliminaries.** A word $w = w_1w_2\dots w_m$ on $\mathbb{N}$ is called a *hook* if $w_1 > w_2$ and either $m = 2$, or $m \geq 3$ and $w_2 < w_3 < \dots < w_m$. As shown in [13], each permutation $\pi = \pi_1\pi_2\dots\pi_n$ admits a unique factorization, called its *hook factorization*, $p\tau_1\tau_2...\tau_r$, where p is an increasing word and each factor $\tau_1, \tau_2, \dots, \tau_k$ is a hook. To derive the hook factorization of a permutation, one can start from the right and factor out each hook step by step. Denote by $\mathrm{inv}(w)$ the numbers of *inversions* of a word $w = w_1w_2\dots w_m$, i.e., the number of pairs (w_i, w_j) such that $i < j$ and $w_i > w_j$. Then we define

$$\mathrm{lec}(\pi) := \sum_{1\leq i\leq k} \mathrm{inv}(\tau_i) \quad \text{and} \quad \mathrm{pix}(\pi) = |p| := \text{length of the factor } p.$$

For example, the hook factorization of $\pi = 1\,3\,4\,14\,12\,2\,5\,11\,15\,8\,6\,7\,13\,9\,10$ is

$$1\,3\,4\,14\,|12\,2\,5\,11\,15\,|8\,6\,7\,|13\,9\,10.$$

Hence $p = 1\,3\,4\,14$, $\tau_1 = 12\,2\,5\,11\,15$, $\tau_2 = 8\,6\,7$, $\tau_3 = 13\,9\,10$, $\mathrm{pix}(\pi) = 4$ and $\mathrm{lec}(\pi) = \mathrm{inv}(12\,2\,5\,11\,15) + \mathrm{inv}(8\,6\,7) + \mathrm{inv}(13\,9\,10) = 7$.

Let $\mathcal{A}_0, \mathcal{A}_1, ..., \mathcal{A}_r$ be a series of sets on $\mathbb{N}$. Denote by $\mathrm{inv}(\mathcal{A}_0, \mathcal{A}_1, ..., \mathcal{A}_r)$ the number of pairs (k, l) such that $k \in \mathcal{A}_i$, $l \in \mathcal{A}_j$, $k > l$ and $i < j$. We usually write $\mathrm{cont}(\mathcal{A})$ the set of all letters in a word $\mathcal{A}$. So we have $(\mathrm{inv} - \mathrm{lec})\pi = \mathrm{inv}(\mathrm{cont}(p), \mathrm{cont}(\tau_1), \ldots, \mathrm{cont}(\tau_r))$ if $p\tau_1\tau_2...\tau_r$ is the hook factorization of π.

From Foata and Han [11, Theorem 1.4], we derive the following combinatorial interpretations of the (q, r)-Eulerian polynomials

$$A_n(t, r, q) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{lec}\pi} r^{\mathrm{pix}\pi} q^{(\mathrm{inv}-\mathrm{lec})\pi}. \tag{2.1}$$

Therefore

$$A_{n,k}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \mathrm{lec}\pi = k}} q^{(\mathrm{inv}-\mathrm{lec})\pi} \quad \text{and} \quad A_{n,k}^{(j)}(q) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \mathrm{lec}\pi = k \\ \mathrm{pix}\pi = j}} q^{(\mathrm{inv}-\mathrm{lec})\pi}. \tag{2.2}$$

It is known [26, Proposition 1.3.17] that the q-binomial coefficient has the interpretation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{(\mathcal{A}, \mathcal{B})} q^{\mathrm{inv}(\mathcal{A}, \mathcal{B})}, \tag{2.3}$$

where the sum is over all ordered partitions $(\mathcal{A}, \mathcal{B})$ of $[n]$ such that $|\mathcal{A}| = k$. We will give bijective proofs of (1.6) and (1.7) using the interpretations in (2.2) and (2.3). To construct our bijective proofs, we need two elementary transformations from [15].

Let τ be a hook with $\mathrm{inv}(\tau) = k$ and $\mathrm{cont}(\tau) = \{x_1, \ldots, x_m\}$, where $x_1 < \ldots < x_m$. Define

$$d(\tau) = x_{m-k+1} x_1 \ldots x_{m-k} x_{m-k+2} \ldots x_m. \tag{2.4}$$

Clearly, $d(\tau)$ is the unique hook with $\mathrm{cont}(d(\tau)) = \mathrm{cont}(\tau)$ and satisfying

$$\mathrm{inv}(d(\tau)) = m - k = |\mathrm{cont}(\tau)| - \mathrm{inv}(\tau).$$

Let τ be a hook or an increasing word with $\mathrm{inv}(\tau) = k$ and $\mathrm{cont}(\tau) = \{x_1, \ldots, x_m\}$, where $x_1 < \ldots < x_m$. Define

$$d'(\tau) = x_{m-k} x_1 \ldots x_{m-k-1} x_{m-k+1} \ldots x_m. \tag{2.5}$$

It is not difficult to see that, $d'(\tau)$ is the unique hook (when $k < m - 1$) or increasing word (when $k = m - 1$) with $\mathrm{cont}(d(\tau)) = \mathrm{cont}(\tau)$ and satisfying

$$\mathrm{inv}(d(\tau)) = m - k - 1 = |\mathrm{cont}(\tau)| - 1 - \mathrm{inv}(\tau).$$

Remark 1. *In [11], a bijection on $\mathfrak{S}_n$ that carries the triplet (fix, exc, maj) to (pix, lec, inv) was constructed without being specified. This bijection consists of two steps. The first step (see [11, section 6]) uses the word analogue of* Kim-Zeng's decomposition *[18] and an updated version of* Gessel-Reutenauer standardization *[14] to construct a bijection on $\mathfrak{S}_n$ that transforms the triplet (fix, exc, maj) to (pix, lec, imaj), where imaj$(\pi) :=$ maj(π^{-1}) for each permutation π. The second step (see [11, section 7]) uses the* Foata's second

fundamental transformation *[8] to carry the triplet* $(pix, lec, imaj)$ *to* (pix, lec, inv). *In view of this bijection, one can construct bijective proofs of* (1.5), (1.6) *and* (1.7) *using the original interpretations in* (1.4), *through the bijective proof of* (1.5) *in [15] and our bijective proofs.*

2.2. **Bijective proof of** (1.6)**.** Let $\mathfrak{S}_n(k) = \{\pi \in \mathfrak{S}_n : \mathrm{pix}(\pi) = k\}$ and $\mathcal{D}_n = \mathfrak{S}_n(0)$. We first notice that the left-hand side of (1.6) has the following interpretation:

$$\sum_{\substack{\pi\in\mathcal{D}_n\\ \mathrm{lec}\pi=a}} q^{(\mathrm{inv}-\mathrm{lec})\pi} = \sum_{k\geq 1}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}A_{k,a}(q). \tag{2.6}$$

This interpretation follows immediately from [24, Corollary 4.4] and (2.1). One can also give a direct combinatorial proof similarly as in [28]. Actually, by (2.2) and (2.3) we have

$$\begin{aligned}
A_{n,a}(q) &= \sum_{\substack{\pi\in\mathfrak{S}_n\\ \mathrm{lec}\pi=a}} q^{(\mathrm{inv}-\mathrm{lec})\pi}\\
&= \sum_k \sum_{\substack{\pi\in\mathfrak{S}_n(k)\\ \mathrm{lec}\pi=a}} q^{\mathrm{inv}(\mathrm{cont}(p),\mathrm{cont}(\tau_1\ldots\tau_r))+\mathrm{inv}(\mathrm{cont}(\tau_1),\mathrm{cont}(\tau_2),\ldots,\mathrm{cont}(\tau_r))}\\
&= \sum_k \sum_{\substack{\mathcal{A}\subseteq[n]\\ |\mathcal{A}|=k}} q^{\mathrm{inv}(\mathcal{A},[n]\setminus\mathcal{A})} \sum_{\substack{\pi\in\mathcal{D}_{n-k}\\ \mathrm{lec}\pi=a}} q^{(\mathrm{inv}-\mathrm{lec})\pi}\\
&= \sum_k \begin{bmatrix}n\\k\end{bmatrix}_q \sum_{\substack{\pi\in\mathcal{D}_k\\ \mathrm{lec}\pi=a}} q^{(\mathrm{inv}-\mathrm{lec})\pi}.
\end{aligned}$$

Applying Gaussian inversion (or q-binomial inversion) to the above identity we obtain (2.6). Now, by (2.6), the symmetrical identity (1.6) is equivalent to the $j=0$ case of the following Lemma.

Lemma 2. *For* $0\leq j\leq n$, *there is an involution* $\mathbf{v}\mapsto\mathbf{u}$ *on* $\mathfrak{S}_n(j)$ *satisfying*

$$lec(\mathbf{u}) = n-j-lec(\mathbf{v}) \quad \text{and} \quad (inv-lec)\mathbf{u} = (inv-lec)\mathbf{v}.$$

Proof. Let $\mathbf{v} = p\tau_1\tau_2\ldots\tau_r$ be the hook factorization of $\mathbf{v}\in\mathfrak{S}_n(j)$, where p is an increasing word and each factor τ_1, τ_2, …, τ_r is a hook. We define $\mathbf{u} = pd(\tau_1)\ldots d(\tau_r)$, where d is defined in (2.4). It is easy to check that this mapping is an involution on $\mathfrak{S}_n(j)$ with the desired properties. □

By (2.2), Lemma 2 gives a simple bijective proof of the following known [5,24] symmetric property of the fix-version of q-Eulerian numbers.

Corollary 3. *For* $n,k,j\geq 0$,

$$A^{(j)}_{n,k}(q) = A^{(j)}_{n,n-j-k}(q). \tag{2.7}$$

2.3. **Bijective proof of** (1.7)**.** Recall [15] that, for a fixed positive integer n, a *two-pix-permutation of* $[n]$ is a sequence of words

$$\mathbf{v} = (p_1, \tau_1, \tau_2, \dots, \tau_{r-1}, \tau_r, p_2) \tag{2.8}$$

satisfying the following conditions:

(C1) p_1 and p_2 are two increasing words, possibly empty;
(C2) $\tau_1, \dots, \tau_r$ are hooks for some positive integer r;
(C3) The concatenation $p_1\tau_1\tau_2\dots\tau_{r-1}\tau_r p_2$ of all components of $\mathbf{v}$ is a permutation of $[n]$.

We also extend the two statistics to the two-pix-permutations by

$$\mathrm{lec}(\mathbf{v}) = \sum_{1\le i\le r} \mathrm{inv}(\tau_i) \quad \text{and} \quad \mathrm{inv}(\mathbf{v}) = \mathrm{inv}(p_1\tau_1\tau_2\dots\tau_{r-1}\tau_r p_2).$$

It follows that

$$(\mathrm{inv}-\mathrm{lec})\mathbf{v} = \mathrm{inv}(\mathrm{cont}(p_1), \mathrm{cont}(\tau_1), \mathrm{cont}(\tau_2), \dots, \mathrm{cont}(\tau_r), \mathrm{cont}(p_2)). \tag{2.9}$$

Let $\mathcal{W}_n(j)$ denote the set of all two-pix-permutations with $|p_1| = j$.

Lemma 4. *Let a, j be fixed nonnegative integers. Then*

$$\sum_{\substack{\mathbf{v}\in\mathcal{W}_n(j)\\ lec\mathbf{v}=a}} q^{(inv-lec)\mathbf{v}} = \sum_{k\ge1} \begin{bmatrix} n\\ k\end{bmatrix}_q A_{n,k}^{(j)}(q). \tag{2.10}$$

Proof. By the hook factorization, the two-pix-permutation in (2.8) is in bijection with the pair (σ, p_2), where $\sigma = p_1\tau_1\tau_2\dots\tau_{r-1}\tau_r$ is a permutation on $[n]\setminus\mathrm{cont}(p_2)$ with $|p_1| = j$ and p_2 is an increasing word. Thus, by (2.2), (2.3) and (2.9), the generating function of all two-pix-permutations $\mathbf{v}$ of $[n]$ with $|p_1| = j$ such that $\mathrm{lec}(\mathbf{v}) = a$ and $|p_2| = n-k$ with respect to the weight $q^{(\mathrm{inv}-\mathrm{lec})\mathbf{v}}$ is $\begin{bmatrix} n\\ n-k\end{bmatrix}_q A_{n,k}^{(j)}(q)$. □

Lemma 5. *Let j be a fixed nonnegative integer. Then there is an involution $\mathbf{v}\mapsto\mathbf{u}$ on $\mathcal{W}_n(j)$ satisfying*

$$lec(\mathbf{v}) = n-j-1-lec(\mathbf{u}), \quad and \quad (inv-lec)\mathbf{v} = (inv-lec)\mathbf{u}.$$

Proof. We give an explicit construction of the bijection using the involutions d and d' defined in (2.4) and (2.5).

Let $\mathbf{v} = (p_1, \tau_1, \tau_2, \dots, \tau_{r-1}, \tau_r, p_2)$ be a two-pix-permutation of $[n]$ with $|p_1| = j$. If $p_2 \neq \emptyset$, then

$$\mathbf{u} = (p_1, d(\tau_1), d(\tau_2), \dots, d(\tau_{r-1}), d(\tau_r), d'(p_2)),$$

otherwise,

$$\mathbf{u} = (p_1, d(\tau_1), d(\tau_2), \dots, d(\tau_{r-1}), d'(\tau_r)).$$

As d and d' are involutions, this mapping is an involution on $\mathcal{W}_n(j)$.

Since we have $\operatorname{lec}(d(\tau_i)) = |\operatorname{cont}(\tau_i)| - \operatorname{lec}(\tau_i)$ for $1 \le i \le r$ and $\operatorname{lec}(d'(p_2)) = |\operatorname{cont}(p_2)| - 1$ in the case $p_2 \neq \emptyset$, it follows that $\operatorname{lec}(\mathbf{u}) = \sum_{i=1}^{r} |\operatorname{cont}(\tau_i)| + |\operatorname{cont}(p_2)| - 1 - \operatorname{lec}(\mathbf{v}) = n - j - 1 - \operatorname{lec}(\mathbf{v})$. The above identity is also valid when $p_2 = \emptyset$.

Finally it follows from (2.9) that $(\operatorname{inv} - \operatorname{lec})\mathbf{u} = (\operatorname{inv} - \operatorname{lec})\mathbf{v}$. This finishes the proof of the lemma. □

Combining Lemmas 4 and 5 we obtain a bijective proof of (1.7).

3. A new recurrence formula for the (q, r)-Eulerian polynomials

The *Eulerian differential operator* δ_x used below is defined by

$$\delta_x(f(x)) := \frac{f(x) - f(qx)}{x},$$

for any $f(x) \in \mathbb{Q}[q][[x]]$ in the ring of formal power series in x over $\mathbb{Q}[q]$ (instead of the traditional $(f(x) - f(qx))/((1 - q)x)$, see [1, 4]). We need the following elementary properties of δ_x.

Lemma 6. *For any $f(x), g(x) \in \mathbb{Q}[q][[x]]$,*

$$\delta_x(f(x)g(x)) = f(qx)\delta(g(x)) + \delta(f(x))g(x)$$

and

$$\delta_x\left(\frac{1}{f(x)}\right) = \frac{-\delta_x(f(x))}{f(qx)f(x)} \quad (f(x) \neq 0).$$

Theorem 7. *The (q, r)-Eulerian polynomials satisfy the following recurrence formula:*

$$A_{n+1}(t, r, q) = rA_n(t, r, q) + tA_n(t, q) + t\sum_{j=1}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q q^j A_j(t, r, q)A_{n-j}(t, q) \tag{3.1}$$

for $n \ge 1$ and $A_1(t, r, q) = r$.

Proof. It is not difficult to show that, for any variable a

$$\delta_z(e(az; q)) = ae(az; q).$$

Now, applying δ_z to both sides of (1.2) and using the above property and Lemma 6, we obtain

$$
\begin{aligned}
&\sum_{n\geq 0} A_{n+1}(t,r,q)\frac{z^n}{(q;q)_n}\\
=&\delta_z\left(\frac{(1-t)e(rz;q)}{e(tz;q)-te(z;q)}\right)\\
=&\delta_z((1-t)e(rz;q))(e(tz;q)-te(z;q))^{-1}+\delta_z\left((e(tz;q)-te(z;q))^{-1}\right)(1-t)e(rzq;q)\\
=&\frac{r(1-t)e(rz;q)}{e(tz;q)-te(z;q)}+\frac{(1-t)e(rzq;q)(te(tz;q)-te(z;q))}{(e(tqz;q)-te(qz;q))(e(tz;q)-te(z;q))}\\
=&r\sum_{n\geq 0}A_n(t,r,q)\frac{z^n}{(q;q)_n}+t\left(\sum_{n\geq 0}A_n(t,r,q)\frac{(qz)^n}{(q;q)_n}\right)\left(\sum_{n\geq 1}A_n(t,q)\frac{z^n}{(q;q)_n}\right).
\end{aligned}
$$

Taking the coefficient of $\frac{z^n}{(q;q)_n}$ in both sides of the above equality, we get (3.1). $\square$

Remark 2. *A different recurrence formula for $A_n(t,r,q)$ was obtained in [24, Corollary 4.3]. Eq.* (3.1) *is similar to two recurrence formulas in the literature: one for the* (inv, des)*-q-Eulerian polynomials in [20, Corollary 2.22] (see also [4]) and the other one for the* (maj, des)*-q-Eulerian polynomials in [20, Corollary 3.6].*

We shall give another interpretation of $A_n(t,r,q)$ in the following.

Let $\pi\in\mathfrak{S}_n$. Recall that an *inversion* of π is a pair $(\pi(i),\pi(j))$ such that $1\leq i<j\leq n$ and $\pi(i)>\pi(j)$. An *admissible inversion* of π is an inversion $(\pi(i),\pi(j))$ that satisfies either

- $1<i$ and $\pi(i-1)<\pi(i)$ or
- there is some l such that $i<l<j$ and $\pi(i)<\pi(l)$.

We write $\mathrm{ai}(\pi)$ the number of admissible inversions of π. Define the statistic $\mathrm{aid}(\pi):=\mathrm{ai}(\pi)+\mathrm{des}(\pi)$. For example, if $\pi=42153$ then there are 5 inversions, but only $(4,3)$ and $(5,3)$ are admissible. So $\mathrm{inv}(\pi)=5$, $\mathrm{ai}(\pi)=2$ and $\mathrm{aid}(\pi)=2+2=4$. The statistics ai and aid are first studied by Shareshian and Wachs [22] in the context of Poset Topology. Here we follow the definitions in [19]. The curious result that the pairs $(\mathrm{aid},\mathrm{des})$ and $(\mathrm{maj},\mathrm{exc})$ are equidistributed on $\mathfrak{S}_n$ was proved in [19] using techniques from poset topology.

Let $\mathcal{W}$ be the set of all the words on $\mathbb{N}$. We define a new statistic, denoted by "rix", on $\mathcal{W}$ recursively. Let $W=w_1w_2\cdots w_n$ be a word in $\mathcal{W}$ and w_i be the rightmost maximum element of W. We define $\mathrm{rix}(W)$ by (with convention that $\mathrm{rix}(\emptyset)=0$)

$$
\mathrm{rix}(W):=\begin{cases}0, & \text{if } i=1\neq n,\\ 1+\mathrm{rix}(w_1\cdots w_{n-1}), & \text{if } i=n,\\ \mathrm{rix}(w_{i+1}w_{i+2}\cdots w_n), & \text{if } 1<i<n.\end{cases}
$$

For example, we have $\mathrm{rix}(1\,5\,2\,4\,3\,3\,5) = 1+\mathrm{rix}(1\,5\,2\,4\,3\,3) = 1+\mathrm{rix}(2\,4\,3\,3) = 1+\mathrm{rix}(3\,3) = 2+\mathrm{rix}(3) = 3$. As every permutation can be viewed as a word on $\mathbb{N}$, this statistic is well-defined on permutations.

We write $\mathfrak{S}_n^{(j)}$ the set of permutations $\pi \in \mathfrak{S}_n$ with $\pi(j) = n$. For $n \geq 1$ and $1 \leq j \leq n$, we define $B_n(t,r,q) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}\,\pi} r^{\mathrm{rix}\,\pi} q^{\mathrm{ai}\,\pi}$ and its restricted version by

$$B_n^{(j)}(t,r,q) := \sum_{\pi\in\mathfrak{S}_n^{(j)}} t^{\mathrm{des}\,\pi} r^{\mathrm{rix}\,\pi} q^{\mathrm{ai}\,\pi}. \tag{3.2}$$

Theorem 8. *We have the following interpretation for (q,r)-Eulerian polynomials:*

$$A_n(t,r,q) = \sum_{\pi\in\mathfrak{S}_n} t^{des\,\pi} r^{rix\,\pi} q^{ai\,\pi}. \tag{3.3}$$

Proof. For $n \geq 1$, it is clear from the definition of $B_n(t,r,q)$ that

$$B_{n+1}(t,r,q) = \sum_{1\leq j\leq n+1} B_{n+1}^{(j)}(t,r,q). \tag{3.4}$$

It is easy to see that

$$B_{n+1}^{(1)}(t,r,q) = tB_n(t,1,q) \quad \text{and} \quad B_{n+1}^{(n+1)}(t,r,q) = rB_n(t,r,q). \tag{3.5}$$

We then consider $B_{n+1}^{(j)}(t,r,q)$ for the case of $1 < j < n+1$.

For a set X, we denote by $\binom{X}{m}$ the m-element subsets of X and $\mathfrak{S}_X$ the set of permutations of X. Let $\mathcal{W}(n,j)$ be the set of all triples (W,π_1,π_2) such that $W \in \binom{[n]}{j}$ and $\pi_1 \in \mathfrak{S}_W, \pi_2 \in \mathfrak{S}_{[n]\setminus W}$. It is not difficult to see that the mapping $\pi \mapsto (W,\pi_1,\pi_2)$ defined by

- $W = \{\pi(i) : 1 \leq i \leq j-1\}$,
- $\pi_1 = \pi(1)\pi(2)\cdots\pi(j-1)$ and $\pi_2 = \pi(j+1)\pi(j+2)\cdots\pi(n)$

is a bijection between $\mathfrak{S}_n^{(j)}$ and $\mathcal{W}(n-1,j-1)$ and satisfies

$$\mathrm{des}(\pi) = \mathrm{des}(\pi_1) + \mathrm{des}(\pi_2) + 1, \quad \mathrm{rix}(\pi) = \mathrm{rix}(\pi_2)$$

and

$$\mathrm{ai}(\pi) = \mathrm{ai}(\pi_1) + \mathrm{ai}(\pi_2) + \mathrm{inv}(W, [n-1]\setminus W) + n - j.$$

Thus, for $1<j<n+1$, we have

$$\begin{aligned}
B_{n+1}^{(j)}(t,r,q) &= \sum_{\pi\in\mathfrak{S}_{n+1}^{(j)}} t^{\mathrm{des}\pi} r^{\mathrm{rix}\pi} q^{\mathrm{ai}\pi}\\
&= tq^{n+1-j} \sum_{(W,\pi_1,\pi_2)\in\mathcal{W}(n,j-1)} q^{\mathrm{inv}(W,[n]\setminus W)} q^{\mathrm{ai}(\pi_1)} t^{\mathrm{des}(\pi_1)} r^{\mathrm{rix}(\pi_2)} q^{\mathrm{ai}(\pi_2)} t^{\mathrm{des}(\pi_2)}\\
&= tq^{n+1-j} \sum_{W\in\binom{[n]}{j-1}} q^{\mathrm{inv}(W,[n]\setminus W)} \sum_{\pi\in\mathfrak{S}_W} q^{\mathrm{ai}(\pi_1)} t^{\mathrm{des}(\pi_1)} \sum_{\pi_2\in\mathfrak{S}_{[n]\setminus W}} r^{\mathrm{rix}(\pi_2)} q^{\mathrm{ai}(\pi_2)} t^{\mathrm{des}(\pi_2)}\\
&= tq^{n+1-j} \begin{bmatrix} n\\ j-1\end{bmatrix}_q B_{j-1}(t,1,q) B_{n+1-j}(t,r,q), \qquad (3.6)
\end{aligned}$$

where we apply (2.3) to the last equality. Substituting (3.5) and (3.6) into (3.4) we obtain

$$B_{n+1}(t,r,q) = rB_n(t,r,q) + tB_n(t,1,q) + t\sum_{j=1}^{n-1} \begin{bmatrix} n\\ j\end{bmatrix}_q q^j B_j(t,r,q) B_{n-j}(t,1,q).$$

By Theorem 7, $B_n(t,r,q)$ and $A_n(t,r,q)$ satisfy the same recurrence formula and initial condition, thus $B_n(t,r,q) = A_n(t,r,q)$. This finishes the proof of the theorem. □

Corollary 9. *The three triplets* (*rix*, *des*, *aid*), (*fix*, *exc*, *maj*) *and* (*pix*, *lec*, *inv*) *are equidistributed on* $\mathfrak{S}_n$.

Remark 3. *At the conference of* Permutation Patterns 2012, *Alexander Burstein [2] gave a direct bijection on* $\mathfrak{S}_n$ *that transforms the triple* (*rix*, *des*, *aid*) *to* (*pix*, *lec*, *inv*)*. The new statistic "rix" was called "aix" instead therein. It would be very interesting to find a similar bijective proof of the equidistribution of* (*rix*, *des*, *aid*) *and* (*fix*, *exc*, *maj*).

4. A symmetrical identity for restricted q-Eulerian numbers

4.1. **An interpretation of $B_{n,k}^{(j)}(q)$ and a proof of Theorem 1.** It follows from (1.2) and (1.8) that $B_{1,0}^{(1)}(q)=1$ and $B_{n,k}^{(1)}(q) = A_{n-1,k-1}(q)$. For $j\ge 2$, we have the following interpretation for $B_{n,k}^{(j)}(q)$.

Lemma 10. *Let* $B_{n,k}^{(j)}(q)$ *be defined by* (1.8)*. For* $j\ge 2$,

$$B_{n,k}^{(j)}(q) = \sum_{\substack{\pi\in\mathfrak{S}_n^{(j)}\\ des(\pi)=k}} q^{ai(\pi)+2j-n-1}.$$

Proof. When $j\ge 2$, by the recurrence relation (3.6), one can compute without difficulty that the exponential generating function $\sum_{n\ge j} q^{2j-n-1} B_n^{(j)}(t,1,q)\frac{z^{n-1}}{(q;q)_{n-1}}$ is exactly the right side of (1.8) using (1.2) and (3.3), which would finish the proof of the lemma. □

Lemma 11. *For $1<j<n$, we have*

$$B_{n,k}^{(j)}(q)=B_{n,n-1-k}^{(j)}(q).$$

Proof. We first construct an involution $f:\pi\mapsto\pi'$ on $\mathfrak{S}_n$ satisfying

$$\mathrm{ai}(\pi)=\mathrm{ai}(\pi') \quad \text{and} \quad \mathrm{des}(\pi)=n-1-\mathrm{des}(\pi'). \tag{4.1}$$

For $n=1$, define $f(\mathrm{id})=\mathrm{id}$. For $n\geq 2$, suppose that $\pi=\pi_1\cdots\pi_n$ is a permutation of $\{\pi_1,\cdots,\pi_n\}$ and π_j is the maximum element in $\{\pi_1,\cdots,\pi_n\}$. We construct f recursively as follows

$$f(\pi)=\begin{cases} f(\pi_2\pi_3\cdots\pi_n)\,\pi_1, & \text{if } j=1,\\ \pi_n\,f(\pi_1\pi_2\cdots\pi_{n-1}), & \text{if } j=n,\\ f(\pi_1\pi_2\cdots\pi_{j-1})\,\pi_j\,f(\pi_{j+1}\pi_{j+2}\cdots\pi_n), & \text{otherwise.}\end{cases}$$

For example, if $\pi=3\,2\,5\,7\,6\,4\,1$, then

$$f(\pi)=f(3\,2\,5)\,7\,f(6\,4\,1)=5\,f(3\,2)7\,f(4\,1)\,6=5\,2\,3\,7\,1\,4\,6.$$

Clearly, $\mathrm{ai}(\pi)=7=\mathrm{ai}(\pi')$ and $\mathrm{des}(\pi)=4=7-1-\mathrm{des}(\pi')$. It is not difficult to see that f is an involution. We can show that f satisfies (4.1) by induction on n, which is routine and left to the reader.

For each $\pi=\pi_1\cdots\pi_{j-1}\,n\,\pi_{j+1}\cdots\pi_n$ in $\mathfrak{S}_n^{(j)}$, we then define

$$g(\pi)=f(\pi_1\cdots\pi_{j-1})\,n\,f(\pi_{j+1}\cdots\pi_n).$$

As f is an involution, g is an involution on $\mathfrak{S}_n^{(j)}$. It follows from (4.1) that $\mathrm{ai}(g(\pi))=\mathrm{ai}(\pi)$ and $\mathrm{des}(\pi)=n-1-\mathrm{des}(g(\pi))$, which completes the proof in view of Lemma 10. □

Remark 4. *A bijective proof of Lemma 11 when $q=1$ was given in [5]. But their bijection does not preserve the admissible inversions. Suppose that $\pi=\pi_1\cdots\pi_n$ is a permutation of $\{\pi_1,\cdots,\pi_n\}$ and π_j is the maximum element in $\{\pi_1,\cdots,\pi_n\}$, we modify f defined above to f' as follows:*

$$f'(\pi)=\begin{cases} f'(\pi_2\pi_3\cdots\pi_n)\,\pi_1, & \textit{if } j=1,\\ \pi, & \textit{if } j=n,\\ f'(\pi_1\pi_2\cdots\pi_{j-1})\,\pi_j\,f'(\pi_{j+1}\pi_{j+2}\cdots\pi_n), & \textit{otherwise.}\end{cases}$$

The reader is invited to check that f' would provide another bijective proof of Corollary 3 using (des, rix, ai).

Now we are in position to give a generating function proof of Theorem 1.

Proof of Theorem 1. We start with the generating function given in (1.8). Multiplying both sides by $e(tz;q)-te(z;q)$, we obtain

$$\sum_{n,k}B_{n,k}^{(j)}(q)t^k\frac{z^{n-1}}{(q;q)_{n-1}}(e(tz;q)-te(z;q))=\frac{(qz)^{j-1}A_{j-1}(t,q)}{(q;q)_{j-1}}(e(tz;q)-te(tz;q)).$$

Expanding the exponential functions, we have

$$\sum_{n,k,i} B^{(j)}_{n,k}(q)\frac{t^{k+i}z^{n+i-1}}{(q;q)_i(q;q)_{n-1}} - \sum_{n,k,i} B^{(j)}_{n,k}(q)\frac{t^{k+1}z^{n+i-1}}{(q;q)_i(q;q)_{n-1}} = \frac{(qz)^{j-1}A_{j-1}(t,q)}{(q;q)_{j-1}}\sum_{n\geq 0}\frac{(1-t)t^n z^n}{(q;q)_n}.$$

Identifying the coefficient of $t^l z^{m-1}$ gives

$$\sum_k \frac{B^{(j)}_{m+k-l,k}(q)}{(q;q)_{l-k}(q;q)_{m+k-l-1}} - \sum_i \frac{B^{(j)}_{m-i,l-1}(q)}{(q;q)_i(q;q)_{m-i-1}} = \frac{q^{j-1}\left(A_{j-1,l+j-m}(q) - A_{j-1,l+j-m-1}(q)\right)}{(q;q)_{j-1}(q;q)_{m-j}}.$$

Multiplying both sides by $(q;q)_{m-1}$, we get

$$\sum_k B^{(j)}_{m+k-l,k}(q)\begin{bmatrix} m-1 \\ l-k \end{bmatrix}_q - \sum_i B^{(j)}_{m-i,l-1}(q)\begin{bmatrix} m-1 \\ i \end{bmatrix}_q = (A_{j-1,l+j-m}(q) - A_{j-1,l+j-m-1}(q))q^{j-1}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q.$$

Changing the variables of the two summations on the left side gives

$$\sum_k B^{(j)}_{k,k+l-m}(q)\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q - \sum_k B^{(j)}_{k,l-1}(q)\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q = (A_{j-1,l+j-m}(q) - A_{j-1,l+j-m-1}(q))q^{j-1}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q. \tag{4.2}$$

We apply the symmetric property in Lemma 11 to the first summation on the left side of (4.2) and we have

$$\sum_k B^{(j)}_{k,k+l-m}(q)\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q = B^{(j)}_{j,j+l-m}(q)\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q + \sum_{k\neq j} B^{(j)}_{k,m-1-l}(q)\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q. \tag{4.3}$$

It follows from Lemma 10 and Theorem 8 that

$$B^{(n)}_{n,k}(q) = \sum_{\substack{\pi\in\mathfrak{S}^{(n)}_n \\ \mathrm{des}(\pi)=k}} q^{\mathrm{ai}(\pi)+n-1} = q^{n-1}A_{n-1,k}(q).$$

Using the symmetric property of $A_{n,k}(q)$, that is $A_{n,k}(q) = A_{n,n-1-k}(q)$, and the above property, the right side of (4.2) can be treated as follows:

$$\begin{aligned}
&(A_{j-1,l+j-m}(q) - A_{j-1,l+j-m-1}(q))q^{j-1}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q \\
=&B^{(j)}_{j,j+l-m}(q)\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q - A_{j-1,m-1-l}(q)q^{j-1}\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q \\
=&B^{(j)}_{j,j+l-m}(q)\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q - B^{(j)}_{j,m-1-l}(q)\begin{bmatrix} m-1 \\ j-1 \end{bmatrix}_q. \qquad (4.4)
\end{aligned}$$

Now we substitute (4.3), (4.4) into (4.2) and obtain

$$\sum_k B^{(j)}_{k,m-1-l}(q)\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q = \sum_k B^{(j)}_{k,l-1}(q)\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q,$$

which becomes (1.9) after setting $m = a+b+2$ and $l-1=b$. $\square$

Remark 5. *The only case that is left out in Theorem 1 is the case of $j=1$. However, as $B^{(1)}_{n,k}(q) = A_{n-1,k-1}(q)$, the corresponding symmetrical identity for this case is* (1.5).

4.2. **Another interpretation of $B^{(j)}_{n,k}(q)$ and a bijective proof of Theorem 1.** Let

$$\bar{\mathfrak{S}}^{(j)}_n := \{\pi \in \mathfrak{S}_n : \pi(j+1) = 1\} \text{ for } 1 \le j < n \quad \text{and} \quad \bar{\mathfrak{S}}^{(n)}_n := \{\pi'\square 1 : \pi' \in \mathfrak{S}_{[n]\setminus\{1\}}\}.$$

The "$\square$" in $\pi = \pi_1\pi_2\cdots\pi_{n-1}\square 1 \in \bar{\mathfrak{S}}^{(n)}_n$ means that the n-th position of π is empty and the hook factorization of π is defined to be $p\tau_1\cdots\tau_r\square 1$, where $p\tau_1\cdots\tau_r$ is the hook factorization of $\pi_1\cdots\pi_{n-1}$ and "$\square 1$" is viewed as a hook. We also define the statistics

$$\operatorname{lec}(\pi_1\pi_2\cdots\pi_{n-1}\square 1) = \sum_{i=1}^r \operatorname{lec}(\tau_i) \quad \text{and} \quad \operatorname{inv}(\pi_1\pi_2\cdots\pi_{n-1}\square 1) = \operatorname{inv}(\pi_1\pi_2\cdots\pi_{n-1}1).$$

For example, $\bar{\mathfrak{S}}^{(3)}_3 = \{32\square 1, 23\square 1\}$ with $\operatorname{lec}(32\square 1) = 1, \operatorname{lec}(23\square 1) = 0$, $\operatorname{inv}(32\square 1) = 3$, and $\operatorname{inv}(23\square 1) = 2$.

Lemma 12. *Let $B^{(j)}_{n,k}(q)$ be defined by* (1.8). *Then $B^{(j)}_{n,k}(q) = \sum_{\substack{\pi\in\bar{\mathfrak{S}}^{(j)}_n \\ lec(\pi)=k}} q^{(inv-lec)\pi}$.*

Proof. Let $\bar{B}^{(j)}_n(t,q) := \sum_{\pi\in\bar{\mathfrak{S}}^{(j)}_n} q^{(\mathrm{inv}-\mathrm{lec})\pi}t^{\mathrm{lec}\,\pi}$. We recall that, to derive the hook factorization of a permutation, one can start from the right and factor out each hook step by step. Therefore, the hook factorization of $\pi = \pi_1\cdots\pi_{j-1}\pi_j 1\pi_{j+2}\cdots\pi_n$ in $\pi \in \bar{\mathfrak{S}}^{(j)}_n$ is $p\tau_1\cdots\tau_s\tau'_1\cdots\tau'_r$, where $p\tau_1\cdots\tau_s$ and $\tau'_1\cdots\tau'_r$ are hook factorizations of $\pi_1\cdots\pi_{j-1}$ and $\pi_j 1\pi_{j+2}\cdots\pi_n$, respectively. When $n > j$, it is not difficult to see that

$$\operatorname{lec}(\pi_j 1\pi_{j+2}\cdots\pi_n) = 1 + \operatorname{lec}(\pi_j\pi_{j+2}\cdots\pi_n)$$

and

$$(\operatorname{inv}-\operatorname{lec})(\pi_j 1\pi_{j+2}\cdots\pi_n) = (\operatorname{inv}-\operatorname{lec})(\pi_j\pi_{j+2}\cdots\pi_n).$$

Thus by (2.3), we have

$$\bar{B}_n^{(j)}(t,q) = A_{j-1}(t,q)q^{j-1}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q tA_{n-j}(t,q) \tag{4.5}$$

for $n > j$. Clearly, $\bar{B}_j^{(j)}(t,q) = A_{j-1}(t,q)q^{j-1}$. So, by (1.2), the exponential generating function $\sum_{n\geq j}\bar{B}_n^{(j)}(t,q)\frac{z^{n-1}}{(q;q)_{n-1}}$ is the right side of (1.8). This finishes the proof of the lemma. □

Remark 6. *This interpretation can also be deduced directly from the interpretation in Lemma 10 using Burstein's bijection [2].*

For $X \subset [n]$ with $|X| = m$ and $1 \in X$, we can define $\bar{\mathfrak{S}}_X^{(j)}$ for $1 \leq j \leq m$ similarly as $\bar{\mathfrak{S}}_m^{(j)}$ like this:

$$\bar{\mathfrak{S}}_X^{(j)} := \{\pi \in \mathfrak{S}_X : \pi(j+1) = 1\} \text{ for } 1 \leq j < m \quad \text{and} \quad \bar{\mathfrak{S}}_X^{(m)} := \{\pi' \Box 1 : \pi' \in \mathfrak{S}_{X\setminus\{1\}}\}.$$

For $1 \leq j \leq n$, we define a *j-restricted two-pix-permutation of* $[n]$ to be a pair $\mathbf{v} = (\pi, p_2)$ satisfying:

- p_2 (possibly empty) is an increasing words on $[n]$ and
- $\pi \in \bar{\mathfrak{S}}_X^{(j)}$ with $X = [n] \setminus \{\mathrm{cont}(p_2)\}$.

Similarly, we define $\mathrm{lec}(\mathbf{v}) = \mathrm{lec}(\pi)$ and $\mathrm{inv}(\mathbf{v}) = \mathrm{inv}(\pi) + \mathrm{inv}(\mathrm{cont}(\pi), \mathrm{cont}(p_2))$. Let $\mathcal{W}_n^{(j)}$ denote the set of all j-restricted two-pix-permutations of $[n]$.

Lemma 13. *Let a, j be positive integers. Then*

$$\sum_{\substack{\mathbf{v}\in\mathcal{W}_n^{(j)} \\ lec\mathbf{v}=a}} q^{(inv-lec)\mathbf{v}} = \sum_{k\geq 1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q B_{k,a}^{(j)}(q). \tag{4.6}$$

Proof. It follows from Lemma 12 and some similar arguments as in the proof of Lemma 4. □

Lemma 14. *Let $2 \leq j \leq n$. Then there is an involution $\mathbf{v} \mapsto \mathbf{u}$ on $\mathcal{W}_n^{(j)}$ satisfying*

$$lec(\mathbf{v}) = n - 2 - lec(\mathbf{u}), \quad and \quad (inv - lec)\mathbf{v} = (inv - lec)\mathbf{u}. \tag{4.7}$$

Proof. Suppose $\mathbf{v} = (\pi, p_2) \in \mathcal{W}_n^{(j)}$ and $\pi = \tau_0\tau_1\cdots\tau_r$ is the hook factorization of π such that τ_0 is a hook or an increasing word and τ_i $(1 \leq i \leq r)$ are hooks. We also assume that $p_2 = x_1 \cdots x_l$ if p_2 is not empty. Note that $1 \notin \mathrm{cont}(\tau_0)$ since $j \neq 1$. We will use the involutions d and d' defined in (2.4) and (2.5). There are several cases to be considered:

(i) $\tau_r = \Box 1$. Then

$$\mathbf{u} = \begin{cases} (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})x_l 1 x_1 x_2 \cdots x_{l-1}, \emptyset), & \text{if } p_2 \neq \emptyset; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})\Box 1, \emptyset), & \text{otherwise.} \end{cases}$$

(ii) $\tau_r = y_s 1 y_1 \cdots y_{s-1}$. Then

$$\mathbf{u} = \begin{cases} (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})d(\tau_r)d'(p_2), \emptyset), & \text{if } p_2 \neq \emptyset; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})\Box 1, y_1\cdots y_s), & \text{if } p_2 = \emptyset \text{ and } y_s > y_{s-1}; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})d'(\tau_r), \emptyset), & \text{otherwise.} \end{cases}$$

(iii) $1 \notin \text{cont}(\tau_r)$. Then

$$\mathbf{u} = \begin{cases} (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})d(\tau_r)d'(p_2), \emptyset), & \text{if } p_2 \neq \emptyset; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1}), d'(\tau_r)), & \text{if } p_2 = \emptyset \text{ and } \text{lec}(\tau_r) = |\tau_r| - 1; \\ (d'(\tau_0)d(\tau_1)\cdots d(\tau_{r-1})d'(\tau_r), \emptyset), & \text{otherwise.} \end{cases}$$

First of all, one can check that $\mathbf{u} \in \mathcal{W}_n^{(j)}$. Secondly, as d, d' are involutions, the above mapping is an involution. Finally, this involution satisfies (4.7) in all cases. This completes the proof of the lemma. $\Box$

Combining Lemmas 13 and 14 we obtain a bijective proof of Theorem 1.

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